LECTURE 34 AREA AND ESTIMATING WITH FINITE SUMS

In the examples, we see that there is always something about "varing" quantities – it is captured by a function $f(x)$. Here, we consider a concrete example about $\int f(x)\,dx$.

Example. Find the area below the graph of the function $f(x) = \sqrt{4 - x^2}$ on $[0, 2]$ by approximating with two vertical bars.

Solution. We know the exact solution here, i.e. the area of a sector with 90 degrees and a radius of 2, $\frac{1}{4} \cdot \pi (2)^2 = \pi \approx 3.14159$.

First, we approximate the area by **two** vertical bars with an identical width of 1, with height decided by the function value on the **left** endpoint. In other words, the first vertical bar takes height $f(0) = 2$ while the second takes $f(1) = \sqrt{3}$. Note that the common width Δx here is simply the length of the interval divided by the number of bars $\frac{2}{2}$. We find that the sum of the area of the two bars is

$$Area = \Delta x \times (f(0) + f(1)) = 1 \times (f(0) + f(1)) = 2 + \sqrt{3} \approx 3.73205.$$

Hmmm. Gross overestimate. How about we now go with **four** bars with identical width of $\frac{1}{2}$? We find

$$Area = \Delta x \times \left(f(0) + f\left(\frac{1}{2}\right) + f(1) + f\left(\frac{3}{2}\right)\right) = \frac{1}{2}\left(2 + \sqrt{3.75} + \sqrt{3} + \sqrt{1.75}\right) \approx 3.49571.$$

Getting better. Let's be finer with **eight** bars with identical width of $\frac{1}{4}$.

$$\begin{aligned} Area &= \Delta x \times \left(f(0) + f\left(\frac{1}{4}\right) + f\left(\frac{1}{2}\right) + f\left(\frac{3}{4}\right) + f(1) + f\left(\frac{5}{4}\right) + f\left(\frac{3}{2}\right) + f\left(\frac{7}{4}\right)\right) \\ &= \frac{1}{4} \times (stuff) \\ &\approx 3.33982. \end{aligned}$$

This is getting much closer than what we started with, relatively speaking (consider the relative error here).

Remark. However, in reality, we don't know the true solution. We are merely refining the partition and getting smaller and smaller numbers. But is this number guaranteed to become static after a large number of refinement? Would it be nice if we can provide a lower bound for the area under the curve, so that we know with the decreasing approximating sum, we won't hit negative infinity. This lower bound can be found by the following procedure, only a slight modification to the above:

Note that the **left endpoint rule** applied to the above problem is always an overestimate, that is, each vertical bar covers the actual area under the curve on its designated subinterval. We call this an **upper sum** as it definitely provides an upper bound for the area. We now proceed to provide a **lower sum** by underestimating the area, with, say, 4 subintervals, which give us points $x = 0, \frac{1}{2}, 1, \frac{3}{2}, 2$. We now pick vertical bars that hide under the curve. For example, for this curve, we choose the **right endpoints**,

$$Area = \Delta x \times \left(f\left(\frac{1}{2}\right) + f(1) + f\left(\frac{3}{2}\right) + f(2)\right) = \frac{1}{2} \times \left(\sqrt{3.75} + \sqrt{3} + \sqrt{1.75} + 0\right) = 2.49571.$$

The overall idea here is to divide and conquer. We here outline **the left endpoint rule**.

(1) First, choose n, a number of subintervals on the given interval $[a, b]$.
(2) Compute width of each subinterval, $\Delta x = \frac{b-a}{n}$.
(3) Label the equidistance points formed by the subintervals, i.e., $x_0 = a, x_1 a + \Delta x, x_2 = a + 2\Delta x, \ldots, x_{n-1} = a + (n-1)\Delta x, x_n = b$. This set of point is called a **partition of** $[a, b]$.
(4) Each bar covers the subinterval $[x_k, x_{k+1}]$ for $k = 0, \ldots, n-1$.
(5) Each bar has a height of $f(x_k)$ (note x_k is the **left** endpoint of the subinterval $[x_k, x_{k+1}]$.)
(6) Total area is
$$\Delta x \cdot (f(x_0) + f(x_1) + f(x_2) + \cdots + f(x_{n-1}) + f(x_n)).$$

This method can be slightly modified at step 5 so that the bar takes height $f(x_{k+1})$ instead. This is called **the right endpoint rule**. If one uses the half way point $f\left(\frac{x_k+x_{k+1}}{2}\right)$, this is called **the midpoint rule**.

The takeaways from the approximation procedure are the following:

(1) There are three basic approximations: **the left endpoint rule, the right endpoint rule, and the midpoint rule.**
(2) To see whether the **left endpoint or right endpoint** rule gives an **upper sum** or **lower sum** depends on the following characteristics of the function
  (a) Increasing/decreasing
  (b) Concavity
(3) **Midpoint rule** gives a sum in-between (not necessarily exactly half way between the other two).

Example. Estimate the total distance travelled by a projectile. The velocity function of a projectile fired straight into the air is $v(t) = 160 - 9.8t$ with unit in meters per second. Use the summation technique just described to estimate how far the projectile rises during the first 3 sec. How close do the sums come to the exact value of 435.9 m?

Solution. Note that distance travelled and displacement are two different things. If the object is always travelling in one direction, that is, has one sign always, then these two notions have the same value. We note that the velocity changes sign at $t = \frac{160}{9.8} \approx 16.33$ which is much beyond 3 seconds. Therefore, in this particular problem, total distance travelled is equal to total displacement.

$$Total\,Displacement = \Delta t\left(v(t_0) + v(t_1) + \cdots + v(t_n)\right)$$

We first consider subintervals of length $\Delta t = 1$ second, chopping the domain $[0,3]$ into $[0,1], [1,2]$ and $[2,3]$. Then, we estimate the total displacement travelled by

(1) Left endpoint rule:

$$\Delta t \cdot (v(0) + v(1) + v(2)) = 1 \cdot [(160 - 9.8 \cdot 0) + (160 - 9.8 \cdot 1) + (160 - 9.8 \cdot 2)] = 450.6.$$

(2) Right endpoint rule:

$$\Delta t \cdot (v(1) + v(2) + v(3)) = 1 \cdot [(160 - 9.8 \cdot 1) + (160 - 9.8 \cdot 2) + (160 - 9.8 \cdot 3)] = 421.2.$$

Remark. Note that our function is a monotone decreasing line. Therefore, left endpoint rule always overestimates, while right endpoint rule always underestimates. This means, the true answer must lie between them.

(3) Midpoint rule:

$$\Delta t \cdot (v(0.5) + v(1.5) + v(2.5)) = 1 \cdot [(160 - 9.8 \cdot 0.5) + (160 - 9.8 \cdot 1.5) + (160 - 9.8 \cdot 2.5)] = 435.9.$$

However, since this is a line, we know its exact area from $[0,3]$. Indeed, it is a trapezoid with upper and lower base 160 and $160 - 9.8 \times 3$, and a height of 3. So, the area is

$$Exact = \frac{1}{2}(160 + 160 - 9.8 \times 3) \times 3 = 435.9.$$

Is it surprising that this coincided with the midpoint rule approximation? It may be at first, since the midpoint rule always gives a bar that overestimate on one side but underestimate on the other. However, the amount of the overestimate and underestimate is exactly equal, and hence offsets each other, only because we are approximating a line.

Now, we distinguish between displacement and distance travelled. Displacement is $\int v(t)\,dt$ where $v(t)$ is velocity. Distance travelled is the absolute displacement at every subinterval, that is,

$$Total\,Distance = \Delta t\left(|v(t_1)| + |v(t_1)| + \cdots + |v(t_n)|\right).$$

Example. Average value of a function. One may think of the arithmetic average as summing all numbers and divide by the number of items. Say,

$$Ave = \frac{1}{n}(x_1 + x_2 + \cdots + x_n).$$

The average value of a continuous function is computed in a similar fashion, though now we replace the sum by an integral, and the "$\frac{1}{n}$" by $\frac{1}{b-a}$ where the average is sought on the interval $[a, b]$.

$$Ave = \frac{1}{b-a}\int f(x)\,dx.$$

Again, we are here required to estimate the integral $\int f(x)\,dx$.